Lesson 15: Related Rates

Important Formulas:

Area of a Square: $A = s^2$
Area of a Circle: $A = \pi r^2$
Circumference of a Circle: $C = 2\pi r$
Surface Area of a Cube: $A = 6s^2$
Volume of a Cube: $V = s^3$

Method:

1. Sketch a picture (if applicable) and find a formula.
2. What rate do you want to find? What do you know?
3. Take the derivative of the formula with respect to t.
4. Plug in your known values.
5. Solve for what you want to find.
6. Answer the question.

1. Find $\frac{d}{dt}(\sin x + e^y - xy)$ if x and y are differentiable functions of t.

$$\frac{d}{dt}[\sin x + e^y - xy] = \boxed{(\cos x)\frac{dx}{dt} + e^y \frac{dy}{dt} - \left[\frac{dx}{dt} y + x\frac{dy}{dt}\right]}$$

2. Assume that x and y are both differentiable functions of t and $x^2y = 2$.

(a) Find $\frac{dx}{dt}$ when $x = 1$ and $\frac{dy}{dt} = 5$.

① $x^2y = 2$

② $x=1$, $\frac{dy}{dt} = 5$, $y = 2$ (plug in $x=1$ to $x^2y=2$)

Want: $\frac{dx}{dt}$

③ $\frac{d}{dt}[x^2y] = \frac{d}{dt}[2]$

$2x\frac{dx}{dt}y + x^2\frac{dy}{dt} = 0$

④ $2(1)\frac{dx}{dt}(2) + (1)^2(5) = 0$

(5/6) $\frac{dx}{dt} = \boxed{\frac{-5}{4}}$

(b) Find $\frac{dy}{dt}$ when $x = 2$ and $\frac{dx}{dt} = -3$.

① $x^2y = 2$

② $x = 2$, $\frac{dx}{dt} = -3$, $y = \frac{1}{2}$ (plug in $x=2$ to $x^2y=2$)

Want: $\frac{dy}{dt}$

③ $\frac{d}{dt}[x^2y] = \frac{d}{dt}[2]$

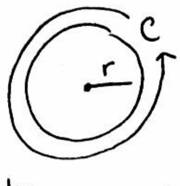
$2x\frac{dx}{dt}y + x^2\frac{dy}{dt} = 0$

④ $2(2)(-3)(\frac{1}{2}) + (2)^2\frac{dy}{dt} = 0$

(5/6) $\frac{dy}{dt} = \frac{6}{4} = \boxed{\frac{3}{2}}$

rate of change

3. If the radius of a circle is shrinking at 3 cm/sec, how quickly is the circumference of the circle shrinking when the radius is 4 cm?

① 

$C = 2\pi r$

② $\frac{dr}{dt} = -3$ (since r is decreasing), $r = 4$

Want: $\frac{dC}{dt}$

③ $\frac{d}{dt}[C] = \frac{d}{dt}[2\pi r]$

$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$

④ $\frac{dC}{dt} = 2\pi(-3)$

⑤ $\frac{dC}{dt} = -6\pi$

⑥ C is decreasing at a rate of $\boxed{6\pi}$ cm/sec

4. If the area of a circle is increasing at π cm^2/sec, how quickly is the radius of the circle changing when the radius is 3cm?

① 

$A = \pi r^2$

② $\frac{dA}{dt} = \pi$, $r = 3$

Want: $\frac{dr}{dt}$

③ $\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2]$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

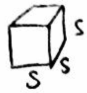
④ $\pi = 2\pi(3)\frac{dr}{dt}$

⑤ $\frac{dr}{dt} = \frac{\pi}{6\pi} = \frac{1}{6}$

⑥ r is changing at a rate of $\boxed{\frac{1}{6}}$ cm/sec

5. If each side of a cubical ice cube is decreasing at a constant rate of 0.5 cm/min,

(a) how quickly is the surface area of the ice cube shrinking when each side is 5cm?

① 

$A = 6s^2$

② $\frac{ds}{dt} = -0.5$ (since s is decreasing), $s = 5$

Want: $\frac{dA}{dt}$

③ $\frac{d}{dt}[A] = \frac{d}{dt}[6s^2]$

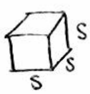
$\frac{dA}{dt} = 12s\,\frac{ds}{dt}$

④ $\frac{dA}{dt} = 12(5)(-0.5)$

⑤ $\frac{dA}{dt} = -30$

⑥ A is shrinking at a rate of $\boxed{30}$ cm²/min

(b) How quickly is the volume of the ice cube shrinking at that moment?

① 

$V = s^3$

② $\frac{ds}{dt} = -0.5, \quad s = 5$ (same as 5a)

Want: $\frac{dV}{dt}$

③ $\frac{d}{dt}[V] = \frac{d}{dt}[s^3]$

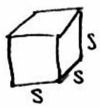
$\frac{dV}{dt} = 3s^2\,\frac{ds}{dt}$

④ $\frac{dV}{dt} = 3(5)^2(-0.5)$

⑤ $\frac{dV}{dt} = -\frac{75}{2}$

⑥ V is shrinking at a rate of $\boxed{\frac{75}{2}}$ cm³/min

6. The volume of a cube is decreasing at 1 in^3/sec. How quickly is the side length of the cube decreasing when the side length is 4 inches?

①  $V = s^3$

② $\frac{dV}{dt} = -1$, $s = 4$

Want: $\frac{ds}{dt}$

③ $\frac{d}{dt}[V] = \frac{d}{dt}[s^3]$

$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$

④ $-1 = 3(4)^2 \frac{ds}{dt}$

⑤ $\frac{ds}{dt} = \frac{-1}{48}$

⑥ s is decreasing at a rate of $\boxed{-\frac{1}{48}}$ in/sec

7. The surface area of a sphere is $A = 4\pi r^2$, where r is the radius of the sphere. The radius is decreasing at a rate of 2 cm/sec. How quickly is the surface area changing when the radius is 3 cm?

①  $A = 4\pi r^2$

② $\frac{dr}{dt} = -2$, $r = 3$

Want: $\frac{dA}{dt}$

③ $\frac{d}{dt}[A] = \frac{d}{dt}[4\pi r^2]$

$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$

④ $\frac{dA}{dt} = 8\pi(3)(-2)$

⑤ $\frac{dA}{dt} = -48\pi$

⑥ A is changing at a rate of $\boxed{-48\pi}$ cm^2/sec

8. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the base and h is the height. A melted ice cream cone is dripping at a rate of 1 cubic ~~cm~~ inch per minute. The height (altitude) of the cone is four times the diameter of the top of the cone. How quickly is the height of ice cream decreasing when there are two inches of ice cream remaining?

① 

$V = \frac{1}{3}\pi r^2 h \leftarrow$ rewrite V in terms of just h:

$V = \frac{1}{3}\pi \left(\frac{1}{8}h\right)^2 h$ since $h = 4(2r) \longrightarrow r = \frac{h}{8}$

$V = \frac{1}{3(8)^2}\pi h^3$

② $\frac{dV}{dt} = -1, \quad h = 2$

Want: $\frac{dh}{dt}$

③ $\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{1}{3(8)^2}\pi h^3\right]$

$\frac{dV}{dt} = \frac{1}{64}\pi h^2 \frac{dh}{dt}$

④ $-1 = \frac{1}{64}\pi (2)^2 \frac{dh}{dt}$

⑤ $\frac{dh}{dt} = -1\left(\frac{64}{4\pi}\right) = -\frac{16}{\pi}$

⑥ h is decreasing at a rate of $\boxed{\frac{16}{\pi}}$ in/min.